SOME IMPLICATIONS OF LEBESGUE DECOMPOSITION

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Abstract. Based on a generalization of Lebesgue decomposition we obtain a characterization of weak compactness in the space $ba(\mathscr{A})$, a representation of its dual space and some results on the structure of finitely additive measures.

1. Introduction and Notation

Throughout the paper Ω will be an arbitrary set, $\mathscr{A}$ an algebra of its subsets, λ a bounded, finitely additive set function on $\mathscr{A}$ (i.e. $\lambda \in ba(\mathscr{A})$) and $\mathscr{M}$ a subset of $ba(\mathscr{A})$.

Among the well known facts of measure theory is the Lebesgue decomposition: for each $\mu \in ba(\mathscr{A})$ there exists s a unique way of writing $\lambda = \lambda_\mu^c + \lambda_\mu^\perp$ where $\lambda_\mu^c \ll \mu$ and $\lambda_\mu^\perp \perp \mu$. In section 2 we prove a slight generalization of this classical result and use it to obtain a characterization of relatively weakly compact subsets of $ba(\mathscr{A})$, in section 3, and on its dual space, in section 4. Some implications of these findings are outlined in section 5. Eventually, in section 6 we exploit Lebesgue decomposition to study some properties of dominated families of finitely additive measures.

The main, simple idea is to treat the orthogonality condition implicit in Lebesgue decomposition as a separating condition for subsets of $ba(\mathscr{A})$, especially in the presence of some form of compactness. A classical result associates relative weak compactness with uniform absolute continuity. In Theorems 1 and 2, we obtain new necessary and sufficient conditions for relative weak compactness of subsets of $ba(\mathscr{A})$ each stating that a corresponding measure theoretic property has to hold uniformly. Following from these, we then obtain, Theorem 3, a complete characterization of the dual space of $ba(\mathscr{A})$ in terms of bounded Cauchy nets. The Riesz representation we propose is unfortunately not as handy as that emerging from the Riesz-Nagy Theorem for Lebesgue spaces. Nevertheless it is helpful in some problems as those treated in Corollary 6. We also exploit it to establish a partial analogue of the Komlós Lemma under finite additivity.

Section 6 considers the absolute continuity property emerging from Lebesgue decomposition and exploits it to investigate some properties of dominated sets of measures. We obtain the finitely additive versions of two classical results, due to Halmos and Savage and to Yan, respectively. Somehow surprisingly, these two Theorems, whose original proofs use countable additivity in an extensive way, carry through unchanged to finite additivity. It is also shown, see Theorem 7, that

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dominated families of set functions have an implicit, desirable property which allows to replace arbitrary families of measurable sets with countable subfamilies.

For the theory of finitely additive measures and integrals we mainly follow the notation and terminology introduced by Dunford and Schwartz [6], although we prefer the symbol $|\lambda|$ to denote the total variation measure generated by λ. $\mathscr{S}(\mathscr{A})$ and $\mathfrak{B}(\mathscr{A})$ designate the families of $\mathscr{A}$ simple functions, endowed with the supremum norm, and its closure, respectively. If $f \in L^1(\lambda)$ we denote its integral interchangeably as $\int f d\lambda$ or $\lambda(f)$ although, when regarded as a set function, we will always use the symbol $\lambda_f \in ba(\mathscr{A})$. We prefer, however, λ_B to $\lambda_{\mathbf{1}_B}$ when $B \in \mathscr{A}$.

We define the following families: $ba(\mathscr{A}, \lambda) = \{\mu \in ba(\mathscr{A}) : \mu \ll \lambda\}$, $ba_1(\mathscr{A}, \lambda) = \{\lambda_f : f \in L^1(\lambda)\}$ and $ba_\infty(\mathscr{A}, \lambda) = \{\mu \in ba(\mathscr{A}) : |\mu| \le c|\lambda| \text{ for some } c > 0\}$ while $\mathbb{P}_{ba}(\mathscr{A})$ will denote the collection of finitely additive probabilities.

The closure of $\mathscr{M}$ in the strong, weak and weak* topology of $ba(\mathscr{A})$ is denoted by $\overline{\mathscr{M}}$, $\overline{\mathscr{M}}^w$ and $\overline{\mathscr{M}}^*$, respectively. We refer to $\mathscr{M}$ the properties holding for each of its elements and use the corresponding symbols accordingly. Thus, we write $\lambda \gg \mathscr{M}$ (resp. $\lambda \perp \mathscr{M}$) whenever $\lambda \gg \mu$ (resp. $\lambda \perp \mu$) for every $\mu \in \mathscr{M}$. $\lambda \gg \mathscr{M}$ is sometimes referred to by saying that $\mathscr{M}$ is dominated by λ.

2. Lebesgue Decomposition

Associated with $\mathscr{M}$ is the collection

$$\mathbf{A}(\mathscr{M}) = \left\{ \sum_n \alpha_n \frac{|\mu_n|}{1 \vee \|\mu_n\|} :\ \mu_n \in \mathscr{M},\ \alpha_n \ge 0 \text{ for } n = 1, 2, \ldots, \sum_n \alpha_n = 1 \right\} \tag{1}$$

as well as the set function

$$\Psi_{\mathscr{M}}(A) = \sup_{\mu \in \mathscr{M}} |\mu|(A) \qquad A \in \mathscr{A} \tag{2}$$

It is at times convenient to investigate the properties of $\mathbf{A}(\mathscr{M})$ rather than $\mathscr{M}$ and we note to this end that $\lambda \gg \mathscr{M}$ (resp. $\lambda \perp \mathscr{M}$) is equivalent to $\lambda \gg \mathbf{A}(\mathscr{M})$ (resp. $\lambda \perp \mathbf{A}(\mathscr{M})$). We say that $\mathscr{M}$ is uniformly absolutely continuous (resp. uniformly orthogonal) with respect to λ, in symbols $\lambda \gg_u \mathscr{M}$ (resp. $\mathscr{M} \perp_u \lambda$) whenever $\lim_{|\lambda|(A) \to 0} \Psi_{\mathscr{M}}(A) = 0$ (resp. when for each ε there exists $A \in \mathscr{A}$ such that $\Psi_{\mathscr{M}}(A) + |\lambda|(A^c) < \varepsilon$). One easily verifies that either of these uniform properties extends from $\mathbf{A}(\mathscr{M})$ to $\mathscr{M}$ if and only if $\mathscr{M}$ is norm bounded.

Lemma 1. *There exists a unique way of writing*

$$\lambda = \lambda^c_{\mathscr{M}} + \lambda^\perp_{\mathscr{M}} \tag{3}$$

where $\lambda^c_{\mathscr{M}}, \lambda^\perp_{\mathscr{M}} \in ba(\mathscr{A})$ are such that (i) $m \gg \lambda^c_{\mathscr{M}}$ for some $m \in \mathbf{A}(\mathscr{M})$ and (ii) $\lambda^\perp_{\mathscr{M}} \perp \mathscr{M}$. If λ is positive or countably additive then so are $\lambda^\perp_{\mathscr{M}}, \lambda^c_{\mathscr{M}}$.

Proof. Take an increasing net $\langle \nu_\alpha \rangle_{\alpha \in \mathfrak{A}}$ in

$$\mathcal{L}(\mathscr{M}) = \{\nu \in ba(\mathscr{A}) : \nu \ll m \text{ for some } m \in \mathbf{A}(\mathscr{M})\} \tag{4}$$

with $\nu = \lim_\alpha \nu_\alpha \in ba(\mathscr{A})$. Extract a sequence $\langle \nu_{\alpha_n} \rangle_{n\in\mathbb{N}}$ such that $\|\nu - \nu_{\alpha_n}\| = (\nu - \nu_{\alpha_n})(\Omega) < 2^{-n-1}$, choose $m_n \in \mathbf{A}(\mathscr{M})$ such that $m_n \gg \nu_{\alpha_n}$ and define $m = \sum_n 2^{-n} m_n \in \mathbf{A}(\mathscr{M})$. Since $m \gg \nu_{\alpha_n}$ for each $n \in \mathbb{N}$ so that there is $\delta_n > 0$ such that $m(A) < \delta_n$ implies $|\nu_{\alpha_n}|(A) < 2^{-n-1}$ and, therefore, $|\nu|(A) \le |\nu_{\alpha_n}|(A) + 2^{-n-1} \le 2^{-n}$. Thus $\mathcal{L}(\mathscr{M})$ is a normal sublattice of $ba(\mathscr{A})$ and (3) is the Riesz decomposition of λ with $\lambda^c_{\mathscr{M}} \in \mathcal{L}(\mathscr{M})$ and $\lambda^\perp_{\mathscr{M}} \perp \mathcal{L}(\mathscr{M})$. □

Of course a different way of stating the same result is the following:

Corollary 1. *Define $\mathcal{L}(\mathscr{M})$ as in (4). Then, $\mathcal{L}(\mathscr{M}) = (\mathscr{M}^\perp)^\perp$.*

Decomposition (3) gains a special interest when combined with some form of compactness.

Lemma 2. *Let $\mathscr{M} \subset ba(\mathscr{A})_+$ be convex and weak* compact. $\lambda \perp \mathscr{M}$ if and only if $\lambda \perp_u \mathscr{M}$.*

Proof. Fix $\varepsilon > 0$ and consider the set

$$\mathcal{K} = \left\{ f \in \mathscr{S}(\mathscr{A}) : 1 \ge f \ge 0,\ |\lambda|(1-f) < \frac{\varepsilon}{4} \right\} \tag{5}$$

If $\lambda \perp \mathscr{M}$, then $\sup_{\mu\in\mathscr{M}} \inf_{f\in\mathcal{K}} \mu(f) < \varepsilon/4$. Endow $ba(\mathscr{A})$ and $\mathscr{S}(\mathscr{A})$ with the weak* and the uniform topology respectively. Then, both $\mathscr{M}$ and $\mathcal{K}$ are convex, the former is compact and the function $\phi(\mu, f) = \mu(f) : ba(\mathscr{A}) \times \mathscr{S}(\mathscr{A}) \to \mathbb{R}$ is separately linear and continuous. By a standard application of Sion's minimax Theorem [9, Corollary 3.3], there exists $f \in \mathcal{K}$ such that $\sup_{\mu\in\mathscr{M}} \mu(f) < \varepsilon/4$. Let $A = \{1 - f < 1/2\} \in \mathscr{A}$. Then Tchebiceff inequality implies $|\lambda|(A^c) + \mu(A) < \varepsilon$ for all $\mu \in \mathscr{M}$. The converse is obvious. □

It is of course possible and perhaps instructive to rephrase the preceding Lemma as a separating condition.

Corollary 2. *Either one of the following mutually exclusive conditions holds: (i) $m \gg \lambda$ for some $m \in \mathbf{A}(\mathscr{M})$ or (ii) there exists $\eta > 0$ such that for each $\mathscr{M}_0 \subset \mathscr{M}$ with $\mathbf{A}(\mathscr{M}_0)$ weak* closed and each $k > 0$ there exists $A \in \mathscr{A}$ for which*

$$|\lambda|(A) > \eta > k\Psi_{\mathbf{A}(\mathscr{M}_0)}(A) \tag{6}$$

If $\mathscr{A}$ is a σ-algebra and $\lambda \in ca(\mathscr{A})$ then (6) rewrites as $|\lambda|(A) > 0 = \Psi_{\mathbf{A}(\mathscr{M}_0)}(A)$ for some $A \in \mathscr{A}$.

Convex, weak* compact subsets of $ba(\mathscr{A})_+$ are often encountered in separation problems, where a family $\mathcal{K}$ of $\mathscr{A}$ measurable functions is given and $\mathscr{M}$ is the set

$$\left\{ m \in \mathbb{P}_{ba}(\mathscr{A}) : \mathcal{K} \subset L(m),\ \sup_{k\in\mathcal{K}} m(k) \le 1 \right\}$$

of separating probabilities. In such special case we learn that if $\lambda \perp \mathscr{M}$ then the two sets $\{\lambda\}$ and $\mathscr{M}$ may be strictly separated by a set in $\mathscr{A}$.

3. The Weak Topology

Decomposition (3) provides some useful insight in the study of weakly compact subsets of $ba(\mathscr{A})$. An exact characterization is the following:

Theorem 1. *If $\mathscr{M}$ is norm bounded the following conditions are equivalent to relative weak compactness:*

(i) $m \gg_u \mathscr{M}$ *for some* $m \in \mathbf{A}(\mathscr{M})$*;*

(ii) the set $\{|\mu| : \mu \in \mathscr{M}\}$ *is uniformly monotone continuous, i.e. if* $\langle A_n \rangle_{n\in\mathbb{N}}$ *is a monotone sequence in* $\mathscr{A}$ *the limit* $\lim_n |\mu|(A_n)$ *exists uniformly in* $\mathscr{M}$*;*

(iii) $\mathscr{M}$ *possesses the uniform orthogonality property, i.e.* $\mathscr{M}_0 \subset \mathscr{M}$ *and* $\lambda \perp \mathscr{M}_0$ *imply* $\lambda \perp_u \mathscr{M}_0$*;*

(iv) for each $\mathscr{M}_0 \subset \mathscr{M}$ *and each sequence* $\langle A_n \rangle_{n\in\mathbb{N}}$ *in* $\mathscr{A}$ *such that*

$$\lim_j \lim_k |\mu| \left(\bigcup_{n=j}^{j+k} A_n \right) = 0 \qquad \mu \in \mathscr{M}_0 \tag{7}$$

$\mu(A_n)$ *converges to* 0 *uniformly with respect to* $\mu \in \mathscr{M}_0$*;*

(v) $\mathscr{M}$ *possesses the uniform absolute continuity property, i.e.* $\mathscr{M}_0 \subset \mathscr{M}$ *and* $\lambda \gg \mathscr{M}_0$ *imply* $\lambda \gg_u \mathscr{M}_0$*.*

Proof. The fact that (i) implies relative weak compactness for norm bounded sets is just [6, IV.9.12].

$(i)\Rightarrow(ii)$. Let $\langle A_n \rangle_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathscr{A}$ and define $\phi_n : ba(\mathscr{A}) \to \mathbb{R}$ by letting

$$\phi_n(\mu) = \lim_k \mu(A_n \cap A_k^c) \qquad \mu \in ba(\mathscr{A}) \tag{8}$$

Then, ϕ_n is continuous and decreases to 0 on the weak closure of $\{|\mu| : \mu \in \mathscr{M}\}$ which, under (i), is compact. By Dini's Theorem, convergence is uniform.

$(ii)\Leftrightarrow(iii)$. Suppose $\lambda \perp \mathscr{M}$ and let $\mathscr{M}_1 = \overline{\mathbf{A}(\mathscr{M})}^{\,*}$. With no loss of generality we can assume $\lambda \geq 0$. We claim that $\lambda \perp \mathscr{M}_1$. If not then there is $m \in \mathscr{M}_1$ such that for some $\eta > 0$ and all $A \in \mathscr{A}$, the inequality $4\eta < m(A) + \lambda(A^c)$ obtains. Fix $m_1 \in \mathbf{A}(\mathscr{M})$ such that $|(m - m_1)(\Omega)| < \eta/2$ and $A_1 \in \mathscr{A}$ such that $m_1(A_1) + \lambda(A_1^c) < \eta$. Assume that $m_1 \ldots, m_{n-1} \in \mathbf{A}(\mathscr{M})$ and $A_1, \ldots, A_{n-1} \in \mathscr{A}$ have been chosen such that

$$m_i(A_i) + \sum_{j\leq i} \lambda(A_j^c) < \eta \quad \text{and} \quad \left| (m_i - m) \left(\bigcap_{j<i} A_j \right) \right| < \eta 2^{-i} \qquad i = 1, \ldots, n-1 \tag{9}$$

Then pick $m_n \in \mathbf{A}(\mathscr{M})$ such that $|(m_n - m)(\bigcap_{j<n} A_j)| < \eta 2^{-n}$ and, by orthogonality, $A_n \in \mathscr{A}$ such that $m_n(A_n) + \lambda(A_n^c) < \eta - \sum_{k=1}^{n-1} \lambda(A_k^c)$. This proves, by induction that it is possible to construct two sequences $\langle m_n \rangle_{n\in\mathbb{N}}$ in $\mathbf{A}(\mathscr{M})$ and $\langle A_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{A}$ that satisfy property (9) for each $n \in \mathbb{N}$. It is then implicit that for all $n, p \in \mathbb{N}$

$$m_n \left(\bigcap_{i=1}^{n+p} A_i \right) + \lambda \left(\bigcup_{i=1}^{n+p} A_i^c \right) \leq m_n(A_n) + \sum_{i=1}^{n} \lambda(A_i^c) + \sum_i \lambda(A_i^c) < 2\eta$$

and so $(m-m_n)\left(\bigcap_{i=1}^{n+p} A_i\right) > 2\eta$. Observe that, under (*ii*), $\mathbf{A}(\mathscr{M})$ is uniformly monotone continuous and so one may fix k sufficiently large so that $\inf_n(m-m_n)\left(\bigcap_{i=1}^{k} A_i\right) > \eta$, contradicting (9). Thus $\lambda \perp \mathscr{M}_1$ and, by Lemma 2, $\lambda \perp \Psi_{\mathscr{M}_1}$ so that $\lambda \perp_u \mathscr{M}$. Given that property (*ii*) extends from $\mathscr{M}$ to each of its subsets, then so does the conclusion just obtained and (*iii*) is proved.

Conversely let $\langle A_n\rangle_{n\in\mathbb{N}}$ be a decreasing sequence in $\mathscr{A}$ so that $\lim_n \lim_k |\mu|(A_n \cap A_k^c) = 0$ for each $\mu \in ba(\mathscr{A})$. Assume the existence of $\varepsilon > 0$ and of a sequence $\langle\mu_n\rangle_{n\in\mathbb{N}}$ in $\mathscr{M}$ such that $\lim_n \lim_k |\mu_n|(A_n \cap A_k^c) > \varepsilon$ and define, similarly to (10),

$$\gamma(A) = \operatorname*{LIM}_n \lim_k \mu_n(A_n \cap A_k^c \cap A) \qquad A \in \mathscr{A}$$

It is obvious that $\gamma(A_n) > \varepsilon$ so that, by Lemma 1, $\gamma_{\mathscr{M}}^{\perp} \neq 0$. However, under (*iii*), $\gamma_{\mathscr{M}}^{\perp} \perp_u \mathscr{M}$ while, by construction, $\gamma \leq \Psi_{\mathscr{M}}$, a contradiction.

(*ii*)⇒(*iv*). Let $\mathscr{M}_0$ and $\langle A_n\rangle_{n\in\mathbb{N}}$ be as in (*iv*). Suppose that, up to the choice of a subsequence, there is ε and a sequence $\langle\mu_n\rangle_{n\in\mathbb{N}}$ in $\mathscr{M}_0$ such that $|\mu_n|(A_n) > \varepsilon$. By (*ii*), for each n there exists $k_n > n$ such that

$$\sup_{\{\mu\in\mathscr{M}_0,\ p\in\mathbb{N}\}} |\mu|\left(\bigcup_{i=n}^{k_n+p} A_i\right) - |\mu|\left(\bigcup_{i=n}^{k_n} A_i\right) < \varepsilon/2$$

Define $\gamma \in ba(\mathscr{A})$ implicitly by setting

$$\gamma(A) = \operatorname*{LIM}_n |\mu_n|(A_n \cap A) \qquad A \in \mathscr{A} \tag{10}$$

where LIM denotes the Banach limit. If $B_j = \bigcup_{i=j}^{k_j} A_i$, one easily concludes

$$\gamma(B_j) = \operatorname*{LIM}_{n>j} |\mu_n|(A_n \cap B_j) > \operatorname*{LIM}_{n>j} |\mu_n|\left(A_n \cap \bigcup_{i=j}^{k_j+n} A_i\right) - \varepsilon/2 = \operatorname*{LIM}_{n>j} |\mu_n|(A_n) - \varepsilon/2 \geq \varepsilon/2$$

while, under (7), $\lim_j |\mu|(B_j) = 0$. By Lemma 1, $\gamma_{\mathscr{M}_0}^{\perp} \neq 0$ and, by (*iii*), $\gamma_{\mathscr{M}_0}^{\perp} \perp_u \mathscr{M}_0$ in contrast with the definition (10).

(*iv*)⇒(*v*). Let $\mathscr{M}_0 \subset \mathscr{M}$ and $\lambda \gg \mathscr{M}_0$. For each $n \in \mathbb{N}$ let $A_n \in \mathscr{A}$ be such that $|\lambda|(A_n) < 2^{-n}$. Then $\sup_k |\lambda|(\bigcup_{i=j}^{k} A_j) < 2^{-j}$ so that, by (*iv*), $|\mu|(A_n)$ converges to 0 uniformly in $\mathscr{M}_0$.

(*v*)⇒(*i*). For each $m \in \mathbf{A}(\mathscr{M})$, let

$$\chi(m) = \sup_{\mu\in\mathscr{M}} \|\mu_m^{\perp}\| \qquad \text{and} \qquad \chi(\mathscr{M}) = \inf_{m\in\mathbf{A}(\mathscr{M})} \chi(m)$$

If $\langle m_n\rangle_{n\in\mathbb{N}}$ is a sequence in $\mathbf{A}(\mathscr{M})$ such that $\chi(m_n) < \chi(\mathscr{M}) + 2^{-n}$ and if we define $m = \sum_n 2^{-n} m_n \in \mathbf{A}(\mathscr{M})$, then from $m \gg m_n$ we conclude $\chi(m) = \chi(\mathscr{M})$. Fix $\gamma_1 = m$ and let $\mu_1 \in \mathscr{M}$ and $A_1 \in \mathscr{A}$ be such that $\gamma_1(A_1) < 2^{-2}$ and $|\mu_1|(A_1) \geq \chi(\mathscr{M})/2$. Assume that $\mu_1, \ldots, \mu_{n-1} \in \mathscr{M}$ and $A_1, \ldots, A_{n-1} \in \mathscr{A}$ have been chosen so that, letting $\gamma_i = \frac{1}{i}(m + |\mu_1| + \ldots + |\mu_{i-1})|$,

$$\gamma_{n-1}(A_{n-1}) < 2^{-2(n-1)} \qquad |\mu_{n-1}|(A_{n-1}) \geq \chi(\mathscr{M})/2 \tag{11}$$

Since $\gamma_n \gg m$, then $\chi(\gamma_n) = \chi(\mathscr{M})$. There exists then $\mu_n \in \mathscr{M}$ such that $\|(\mu_n)_{\gamma_n}^{\perp}\| > \chi(\mathscr{M})/2$ and thus a set $A_n \in \mathscr{A}$ such that $|\mu_n|(A_n) \geq \chi(\mathscr{M})/2$ while $\gamma(A_n) < 2^{-2n}$. It follows by induction that

there are sequences $\langle\mu_n\rangle_{n\in\mathbb{N}}$ and $\langle A_n\rangle_{n\in\mathbb{N}}$ such that for all n

$$\sup_{i<n}|\mu_i|(A_{n-1})<2^{-n}\quad\text{and}\quad|\mu_n|(A_n)\geq\chi(\mathscr{M})/2$$

Let $\mu=\sum_n 2^{-n}|\mu_n|$. (v) implies that the sequence $\langle\mu_n\rangle_{n\in\mathbb{N}}$ is uniformly absolutely continuous with respect to μ; on the other hand,

$$\mu(A_k)=\sum_n 2^{-n}|\mu_n|(A_k)\leq\sum_{n=1}^{k}2^{-n}|\mu_n|(A_k)+2^{-k}\leq 2^{-(k-1)}$$

so that $\chi(\mathscr{M})\leq 2\lim_k|\mu_k|(A_k)=0$. But then $\chi(m)=0$ i.e. $m\gg\mathscr{M}$ and, by (v), $m\gg_u\mathscr{M}$. $\square$

We also conclude

Corollary 3. *Let $\mathscr{M}\subset ba(\mathscr{A})$ be relatively weakly compact. Then, (i) $\lambda\perp\mathscr{M}$ if and only if $\lambda\perp_u\overline{\mathbf{A}(\mathscr{M})}^{\,*}$ and (ii) $m\in\overline{\mathbf{A}(\mathscr{M})}^{\,*}$ implies $m^{\perp}_{\mathscr{M}}=0$.*

Proof. In the proof of the implication $(ii)\Rightarrow(iii)$ of Theorem 1 we showed that $\lambda\perp\mathscr{M}$ if and only if $\lambda\perp\overline{\mathbf{A}(\mathscr{M})}^{\,*}$. (i) then follows from Corollary 6; the second from (i) and Lemma 1. $\square$

Theorem 1 has a number of implications which help clarifying the relationship with other well known criteria for relative weak compactness. For example, $\mathscr{M}$ is relatively weakly compact if and only if $\{|\mu|:\mu\in\mathscr{M}\}$ is so. Moreover, all disjoint sequences of sets satisfy condition (7) (by boundedness) so that if $\mathscr{M}$ is relatively weakly compact then necessarily $m(A_n)$ converges to 0 uniformly in $\mathscr{M}$ for every disjoint sequence, a property of weakly convergent sequences already outlined in [2, Theorem 8.7.3]. Thus, Theorem 1 contains [11, Theorem 1.3] as a special case. Another immediate consequence is that a subset of $ca(\mathscr{A})$ is relatively weakly compact if and only if norm bounded and uniformly countably additive or, equivalently, uniformly absolutely continuous with respect to some $\lambda\in ca(\mathscr{A})$, see [6, IV.9.1 and IV.9.2].

Another characterization of weak compactness is given in the following Theorem 2. A sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $\mathscr{S}(\mathscr{A})$ is said to be uniformly bounded whenever $\sup_n\|f_n\|<\infty$.

Theorem 2. *In the following, conditions (i)–(ii) are equivalent and imply (iii):*

(i) $\mathscr{M}$ is relatively weakly compact;

(ii) $\mathscr{M}$ is bounded and possesses the uniform Cauchy property, i.e. if $\mathscr{M}_0\subset\mathscr{M}$ and $\langle f_n\rangle_{n\in\mathbb{N}}$ is a uniformly bounded sequence in $\mathscr{S}(\mathscr{A})$ which is Cauchy in $L^1(\mu)$ for all $\mu\in\mathscr{M}_0$, then

$$\lim_n\sup_{\mu\in\mathscr{M}_0}\sup_{p,q}|\mu|(|f_{n+p}-f_{n+q}|)=0\tag{12}$$

(iii) $\mathscr{M}$ is bounded and for each sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ as in (ii) and each sequence $\langle\mu_k\rangle_{k\in\mathbb{N}}$ in $\mathscr{M}$

$$\operatorname*{LIM}_k\lim_n\mu_k(f_n)=\lim_n\operatorname*{LIM}_k\mu_k(f_n)\tag{13}$$

Proof. $(i)\Rightarrow(ii)$ If $\mathscr{M}$ is relatively weakly compact it is bounded and uniformly absolutely continuous with respect to some $m \in \mathbf{A}(\mathscr{M})$. If $\langle f_n\rangle_{n\in\mathbb{N}}$ is uniformly bounded and Cauchy in $L^1(\mu)$ for all $\mu \in \mathscr{M}$, then it is Cauchy in $L^1(m)$ too. Moreover, given that

$$|\mu|(|f_{k+p} - f_{k+q}|) \leq 2\sup_n \|f_n\| \; |\mu|^*(|f_{k+p} - f_{k+q}| \geq c) + c \sup_{\mu\in\mathscr{M}} \|\mu\|$$

(12) follows from uniform absolute continuity.

$(ii)\Rightarrow(iii)$ It follows from the inequality

$$\left|\operatorname*{LIM}_k \lim_n \mu_k(f_n) - \lim_i \operatorname*{LIM}_k \mu_k(f_i)\right| \leq \lim_n \sup_k \sup_{p,q} |\mu_k|(|f_{n+p} - f_{n+q}|)$$

$(ii)\Rightarrow(i)$ Choose the sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in (ii) to consist of indicators of a decreasing sequence $\langle A_n\rangle_{n\in\mathbb{N}}$ of $\mathscr{A}$ measurable sets. Then (12) implies that $\{|\mu| : \mu \in \mathscr{M}\}$ is uniformly monotone continuous. □

4. The Representation of Continuous Linear Functionals on $ba(\mathscr{A})$

The class of sequences introduced in Theorem 2 will be in this section the basis to obtain a representation of continuous linear functionals on $ba(\mathscr{A})$. To this end we need some additional notation. $f \in \mathfrak{B}(\mathscr{A})$ and $\mu \in ba(\mathscr{A})$ admit the Stone space representation as $\tilde{f} \in \mathcal{C}(\tilde{\mathscr{A}})$ and $\tilde{\lambda} \in ca(\sigma\tilde{\mathscr{A}})$ where $\tilde{\mathscr{A}}$ is the algebra of all clopen sets of a compact, Hausdorff, totally disconnected space $\tilde{\Omega}$ such that $\mu(f) = \tilde{\mu}(\tilde{f})$, [6].

In the following we also use $\mathscr{L}(\mathscr{A})$ for the space of continuous linear operators $T : ba(\mathscr{A}) \to ba(\mathscr{A})$ and $\mathscr{L}_*(\mathscr{A})$ for the subspace of those $T \in \mathscr{L}(\mathscr{A})$ possessing the additional property

$$T(\mu_f) = T(\mu)_f \qquad f \in L^1(\mu), \mu \in ba(\mathscr{A}) \tag{14}$$

Remark that if $A, A_1, \ldots, A_N \in \mathscr{A}$ with $A_n \cap A_m = \varnothing$ for $n \neq m$, then (14) implies

$$\sum_{n=1}^N |T(\mu)(A\cap A_n)| = \sum_{n=1}^N |T(\mu_{A\cap A_n})(\Omega)| \leq \|T\| \sum_{n=1}^N \|\mu_{A\cap A_n}\| = \|T\| \sum_{n=1}^N |\mu|(A\cap A_n) \leq \|T\||\mu|(A)$$

so that $|T(\mu)| \leq \|T\||\mu|$, i.e. $T(\mu) \in ba_\infty(\mathscr{A}, \mu)$. Eventually, if $T \in \mathscr{L}(\mathscr{A})$ let T_λ denote its restriction to $ba(\mathscr{A}, \lambda)$.

Proposition 1. *$ba(\mathscr{A})^*$ is isometrically isomorphic to the space $\mathscr{L}_*(\mathscr{A})$ and the corresponding elements are related via the identity*

$$\phi(\mu) = T(\mu)(\Omega) \qquad \mu \in ba(\mathscr{A}) \tag{15}$$

Moreover, there is a sequence $\langle f_n^\lambda\rangle_{n\in\mathbb{N}}$ in $\mathscr{S}(\mathscr{A})$ uniformly bounded by $\|T_\lambda\|$ which is Cauchy in $L^1(\mu)$ for all $\mu \in ba(\mathscr{A}, \lambda)$ and such that

$$\limsup_n \|f_n^\lambda\| = \|T_\lambda\| \quad \textit{and} \quad T(\mu) = \lim_n \mu(f_n^\lambda) \qquad \mu \in ba(\mathscr{A}, \lambda) \tag{16}$$

If T is positive then $\langle f_n^\lambda\rangle_{n\in\mathbb{N}}$ can be chosen to be positive.

Proof. If $T \in \mathscr{L}_*(\mathscr{A})$ it is obvious that the right hand side of (16) implicitly defines a continuous linear functional on $ba(\mathscr{A})$ and that $\|\phi\| \le \|T\|$. Conversely, let $\phi \in ba(\mathscr{A})^*$, fix $\mu \in ba(\mathscr{A})$ and define the set function $T(\mu)$ on $\mathscr{A}$ implicitly by letting

$$T(\mu)(A) = \phi(\mu_A) \qquad A \in \mathscr{A} \tag{17}$$

$T(\mu)$ is additive by the linearity of ϕ. Moreover, if $A_1, \ldots, A_N \in \mathscr{A}$ are disjoint then

$$\sum_{n=1}^N |T(\mu)(A \cap A_n)| = \sum_{n=1}^N |\phi(\mu_{A\cap A_n})| \le \sum_{n=1}^N \|\phi\|\|\mu_{A\cap A_n}\| = \sum_{n=1}^N \|\phi\||\mu|(A\cap A_n) = \|\phi\||\mu|(A)$$

so that $|T(\mu)| \le \|\phi\||\mu|$. It follows that $T(\mu) \in ba_\infty(\mathscr{A}, \mu)$ and $\|T\| \le \|\phi\|$. Since $(\mu_A)_B = \mu_{A\cap B}$, we conclude from (17) that $T(\mu)(A \cap B) = T(\mu_A)(B)$ so that $T(\mu_A) = T(\mu)_A$ for all $A \in \mathscr{A}$. This conclusion extends by linearity to $\mathscr{S}(\mathscr{A})$. If $\langle f_n \rangle_{n\in\mathbb{N}}$ is a fundamental sequence for $f \in L^1(\mu) \subset L^1(T(\mu))$, then by continuity

$$T(\mu_f) = \lim_n T(\mu_{f_n}) = \lim_n T(\mu)_{f_n} = T(\mu)_f$$

and we conclude that $T \in \mathscr{L}_*(\mathscr{A})$. (16) thus defines a linear isometry of $\mathscr{L}_*(\mathscr{A})$ onto $ba(\mathscr{A})^*$. To conclude that this is an isomorphism let $T_1, T_2 \in \mathscr{L}_*(\mathscr{A})$ and let ϕ_1, ϕ_2 be the associated elements of $ba(\mathscr{A})^*$. If $T_1 \ne T_2$ then $T_1(\mu) \ne T_2(\mu)$ for some $\mu \in ba(\mathscr{A})$ and thus, by (17), $\phi_1(\mu_A) = T_1(\mu)(A) \ne T_2(\mu)(A) = \phi_2(\mu_A)$ for some $A \in \mathscr{A}$.

To prove (16), denote by $\sigma : ba(\mathscr{A}) \to ca(\sigma\tilde{\mathscr{A}})$ the Stone isomorphism. Then, if $T \in \mathscr{L}_*(\mathscr{A})$ and $\tilde{T} = \sigma \cdot T\sigma^{-1}$ one immediately concludes that $\tilde{T} : ca(\sigma\tilde{\mathscr{A}}) \to ca(\sigma\tilde{\mathscr{A}})$ and that

$$\tilde{T}(\tilde{\mu}_{\tilde{f}}) = \lim_n \tilde{T}(\tilde{\mu}_{\tilde{f}_n}) = \lim_n \sigma\left(T(\mu_{f_n})\right) = \lim_n \sigma\left(T(\mu)_{f_n}\right) = \lim_n \sigma\left(T(\mu)\right)_{\tilde{f}_n} = \lim_n \tilde{T}(\tilde{\mu})_{\tilde{f}_n} = \tilde{T}(\tilde{\mu})_{\tilde{f}}$$

so that $\tilde{T} \in \mathscr{L}_*(\sigma\tilde{\mathscr{A}})$. Exploiting the existence of Radon Nikodym derivatives we conclude that when $\mu \in ba(\mathscr{A}, \lambda)$,

$$\tilde{T}(\tilde{\mu}) = \tilde{T}(\tilde{\lambda})_{\tilde{f}^\mu} = \int \tilde{f}^\mu \tilde{f}^\lambda d|\tilde{\lambda}| = \int \tilde{f}^\lambda d\tilde{\mu}$$

with $\tilde{f}^\mu \in L^1(\tilde{\lambda})$, $\tilde{f}^\lambda \in L^\infty(\tilde{\lambda})$ and $\|\tilde{f}^\lambda\|_{L^\infty} \le \|T_\lambda\|$. Let, as usual,

$$\tilde{f}_n^\lambda = \sum_{i=-2^n}^{2^n} i2^{-n}\|T_\lambda\|\mathbf{1}_{\{i2^{-n}\|T_\lambda\| \le \tilde{f}^\lambda < (i+1)2^{-n}\|T_\lambda\|\}}$$

The sequence $\left\langle \tilde{f}_n^\lambda \right\rangle_{n\in\mathbb{N}}$ in $\mathscr{S}(\sigma\tilde{\mathscr{A}})$ is increasing, converges uniformly to $\tilde{f}^\lambda$ and is positive if T_λ is so. Replacing each $\sigma\tilde{\mathscr{A}}$ measurable set in the support of $\tilde{f}_n^\lambda$ with a corresponding $\tilde{\mathscr{A}}$ measurable set arbitrarily close to it in $\tilde{\lambda}$ measure, we obtain a sequence $\left\langle \hat{f}_n^\lambda \right\rangle_{n\in\mathbb{N}}$ in $\mathscr{S}(\tilde{\mathscr{A}})$ such that (i) $\|\hat{f}_n^\lambda\| \le \|T_\lambda\|$, (ii) $\hat{f}_n^\lambda$ is positive if T_λ is so and (iii) $\left\langle \hat{f}_n^\lambda \right\rangle_{n\in\mathbb{N}}$ converges to $\tilde{f}^\lambda$ in $L^1(\tilde{\mu})$ for each $\mu \in ba(\mathscr{A}, \lambda)$, by [6, III.3.6]. Let $f_n^\lambda = \sigma^{-1}\left(\hat{f}_n^\lambda\right) \in \mathscr{S}(\mathscr{A})$. Then,

$$T(\mu) = \sigma^{-1}\left(\tilde{T}(\tilde{\mu})\right) = \lim_n \sigma^{-1}\left(\int \hat{f}_n^\lambda d\tilde{\mu}\right) = \lim_n \int \sigma^{-1}\left(\hat{f}_n^\lambda\right) d\mu = \lim_n \int f_n^\lambda d\mu \tag{18}$$

so that $\|T_\lambda\| \le \limsup_n \|f_n^\lambda\|$. Properties (i) and (ii) carry over to the sequence $\langle f_n^\lambda \rangle_{n\in\mathbb{N}}$, by the properties of the Stone isomorphism, and therefore $\|T_\lambda\| = \limsup_n \|f_n^\lambda\|$. Moreover,

$$\lim_n \sup_{p,q} |\mu| \left(\left| f_{n+p}^\lambda - f_{n+q}^\lambda \right| \right) = \lim_n \sup_{p,q} |\tilde{\mu}| \left(\left| \hat{f}_{n+p}^\lambda - \hat{f}_{n+q}^\lambda \right| \right) = 0$$

so that the sequence is Cauchy in $L^1(\mu)$ for all $\mu \in ba(\mathscr{A}, \lambda)$. □

Implicit in Proposition 1 is a simple proof of the following, important result.

Corollary 4 (Berti and Rigo). *The dual space of $L^1(\lambda)$ is isomorphic to $ba_\infty(\mathscr{A}, \lambda)$ and the corresponding elements are related via the identity*

$$\varphi(f) = \mu(f) \qquad f \in L^1(\lambda) \tag{19}$$

Proof. By the isometric isomorphism between $L^1(\lambda)$ and $ba_1(\mathscr{A}, \lambda)$ and Proposition 1, each continuous linear functional φ on $L^1(\lambda)$ corresponds isometrically to some $T \in \mathscr{L}_*(\mathscr{A})$ via the identity $\varphi(f) = T(\lambda)(f)$. Write $\mu = T(\lambda)$. Conversely, if $\mu \in ba_\infty(\mathscr{A}, \lambda)$ then it is obvious the right hand side of (19) defines a continuous linear functional on $L(\lambda)$. □

Another interesting conclusion is

Corollary 5. *For every uniformly bounded net $\langle h_\alpha \rangle_{\alpha\in\mathfrak{A}}$ in $\mathfrak{B}(\mathscr{A})$ there exists a uniformly bounded sequence $\langle f_n \rangle_{n\in\mathbb{N}}$ in $\mathscr{S}(\mathscr{A})$ which is Cauchy in $L^1(\mu)$ for all $\mu \in ba(\mathscr{A}, \lambda)$ and such that*

$$\operatorname*{LIM}_a \mu(h_a \mathbf{1}_A) = \lim_n \mu(f_n \mathbf{1}_A) \qquad A \in \mathscr{A},\ \mu \in ba(\mathscr{A}, \lambda) \tag{20}$$

If $\langle h_\alpha \rangle_{\alpha\in\mathfrak{A}}$ is increasing then $\langle f_n \rangle_{n\in\mathbb{N}}$ can be chosen to be increasing too.

Proof. The existence claim follows from Proposition 1 upon noting that the left hand side of (20) indeed defines a continuous linear functional on $ba(\mathscr{A})$. □

Corollary 5 suggests that dominated families of measures admit an implicit, denumerable structure. This intuition will be made precise in the next section.

An exact integral representation of the form $\phi(\mu) = \mu(f)$ for elements of $ba(\mathscr{A})$ is not possible in general, see [2, 9.2.1]. On the other hand, the representation (16) may seem unsatisfactory inasmuch the intervening sequence depends on the choice of λ. This last remark also applies to $ca(\mathscr{A})$, a space for which, despite the characterization of weak compactness, a representation of continuous linear functionals is missing. The following result provides an answer.

Theorem 3. *A linear functional ϕ on $ba(\mathscr{A})$ is continuous if and only if it admits the representation*

$$\phi(\mu) = \lim_\alpha \mu(f_\alpha) \qquad \mu \in ba(\mathscr{A}) \tag{21}$$

where $\langle f_\alpha \rangle_{\alpha\in\mathfrak{A}}$ is a uniformly bounded net in $\mathscr{S}(\mathscr{A})$ with $\limsup_{\alpha\in\mathfrak{A}} \|f_\alpha\| = \|\phi\|$ which is Cauchy in $L^1(\mu)$ for all $\mu \in ba(\mathscr{A})$.

Proof. It is easily seen that if the net $\langle f_\alpha\rangle_{\alpha\in\mathfrak{A}}$ is as in the claim, then the right hand side of (21) indeed defines a continuous linear functional on $ba(\mathscr{A})$ and that $\|\phi\|\le\limsup_\alpha\|f_\alpha\|$. For the converse, passing to the Stone space representation and given completeness of $ca(\sigma\tilde{\mathscr{A}})$, (16) becomes

$$T(\mu)(A)=\tilde\mu(\mathbf{1}_{\tilde A}\tilde f^\lambda)\qquad A\in\mathscr{A},\ \mu\in ba(\mathscr{A},\lambda)\tag{22}$$

for some $\tilde f^\lambda\in L^\infty(\tilde\lambda)$ with $|\tilde f^\lambda|\le\|T_\lambda\|$. Let $\mathfrak{A}$ be the collection of all finite subsets of $ba(\mathscr{A})$ directed by inclusion. For each $\alpha\in\mathfrak{A}$ choose $\lambda_\alpha\in ba(\mathscr{A})$ such that $\lambda_\alpha\gg\alpha$. Of course, for each $\mu\in ba(\mathscr{A})$ there exists $\alpha\in\mathfrak{A}$ such that $\lambda_\alpha\gg\mu$. We then get the representation

$$T(\mu)(A)=\tilde\mu(\tilde f^{\lambda_\alpha}\mathbf{1}_{\tilde A})\qquad A\in\mathscr{A},\ \mu\in\alpha,\ \alpha\in\mathfrak{A}\tag{23}$$

with $\|\tilde f^{\lambda_\alpha}\|\le\|T_{\lambda_\alpha}\|$. Fix $\tilde f_\alpha\in\mathscr{S}(\tilde{\mathscr{A}})$ such that $\|\tilde f_\alpha\|\le\|\tilde f^{\lambda_\alpha}\|$ and

$$\sup_{\mu\in\alpha}\tilde\mu\left(\left|\tilde f^{\lambda_\alpha}-\tilde f_\alpha\right|\right)\le 2^{-|\alpha|-1}$$

and let $f_\alpha\in\mathscr{S}(\mathscr{A})$ correspond to $\tilde f_\alpha$ under the Stone isomorphism. Then,

$$\lim_\alpha\mu(f_\alpha\mathbf{1}_A)=\lim_\alpha\tilde\mu(\tilde f_\alpha\mathbf{1}_{\tilde A})=\lim_\alpha\tilde\mu(\tilde f^{\lambda_\alpha}\mathbf{1}_{\tilde A})=T(\mu\mathbf{1}_A)\qquad A\in\mathscr{A},\ \mu\in ba(\mathscr{A})$$

which, together with (21), proves the existence of the representation (16) and of the inequality $\limsup_\alpha\|f_\alpha\|\le\lim_\alpha\|T_{\lambda_\alpha}\|\le\|T\|=\|\phi\|$. Moreover, if $\alpha_1,\alpha_2,\alpha\in\mathfrak{A}$ and $\mu\in\alpha\subset\alpha_1,\alpha_2$, then

$$\begin{aligned}\mu(|f_{\alpha_1}-f_{\alpha_2}|)&=\mu\left(h(\alpha_1,\alpha_2)(f_{\alpha_1}-f_{\alpha_2})\right)\\&\le 2^{-|\alpha|}+\tilde\mu\left(\tilde h(\alpha_1,\alpha_2)(\tilde f^{\lambda_{\alpha_1}}-\tilde f^{\lambda_{\alpha_2}})\right)\\&=2^{-|\alpha|}\end{aligned}$$

the third line following from (23) and the inclusion $h(\alpha_1,\alpha_2)\in\mathscr{S}(\mathscr{A})$. But then $\langle f_\alpha\rangle_{\alpha\in\mathfrak{A}}$ is indeed a Cauchy net in $L^1(\mu)$ for all $\mu\in ba(\mathscr{A})$. □

The space of uniformly bounded nets in $\mathscr{S}(\mathscr{A})$ is a linear space if, for $\tilde f=\langle f_\alpha\rangle_{\alpha\in\mathfrak{A}}$ and $\tilde g=\langle g_\delta\rangle_{\delta\in\mathfrak{D}}$ two such nets, we endow $\mathfrak{A}\times\mathfrak{D}$ with the product order obtained by letting $(\alpha_1,\delta_1)\ge(\alpha_2,\delta_2)$ whenever $\alpha_1\ge\alpha_2$ and $\delta_1\ge\delta_2$ and write $\tilde f+\tilde g$ as $\langle f_\alpha+g_\delta\rangle_{(\alpha,\delta)\in\mathfrak{A}\times\mathfrak{D}}$. Theorem 3 suggests the definition of a seminorm on such space by letting

$$\|F\|=\limsup_\alpha\|f_\alpha\|\qquad\text{whenever}\qquad F=\langle f_\alpha\rangle_{\alpha\in\mathfrak{A}}\tag{24}$$

and denote by $\mathfrak{C}(\mathscr{A})$ the linear space of equivalence classes of uniformly bounded nets in $\mathscr{S}(\mathscr{A})$ which are Cauchy in $L^1(\mu)$ for all $\mu\in ba(\mathscr{A})$.

Theorem 4. *The identity* (21) *defines an isometric isomorphism between $ba(\mathscr{A})^*$ and $\mathfrak{C}(\mathscr{A})$.*

Proof. The right hand side of (21) is invariant upon replacing the net $F=\langle f_\alpha\rangle_{\alpha\in\mathfrak{A}}$ with $G=\langle g_\delta\rangle_{\delta\in\mathfrak{D}}$ whenever $\|F-G\|=0$. □

5. Some Implications.

The characterization so obtained is admittedly not an easy one, due to the intrinsic difficulty of identifying explicitly the net associated to each continuous functional. It has, this notwithstanding, a number of interesting implications. We illustrate some with no claim of completeness.

Corollary 6. *Let $\mathscr{M}$ and $\mathscr{N}$ be convex, weakly compact subsets of $ba(\mathscr{A})$. Then,*

(i) $\mathscr{M} \cap \mathscr{N} = \varnothing$ if and only if there exists $f \in \mathscr{S}(\mathscr{A})$ such that $\inf_{\nu \in \mathscr{N}} \nu(f) > \sup_{\mu \in \mathscr{M}} \mu(f)$;

(ii) there exists $\mathcal{K} \subset \mathscr{S}(\mathscr{A})$ and a subset $\mathscr{M}_0$ of extreme points of $\mathscr{M}$ such that

$$\mathscr{M} = \left\{ m \in ba(\mathscr{A}) : m(k) \le \max_{\mu \in \mathscr{M}_0} \mu(k) \text{ for all } k \in \mathcal{K} \right\} \tag{25}$$

Proof. (i). The weak topology is linear. There is then a linear functional ϕ on $ba(\mathscr{A})$ and $a, b \in \mathbb{R}$ such that $\inf_{\nu \in \mathscr{N}} \phi(\nu) > a > b > \sup_{\mu \in \mathscr{M}} \phi(\mu)$. By compactness, $\mathscr{M}$ and $\mathscr{N}$ are dominated so that, by Proposition 1, ϕ is associated with a uniformly bounded, Cauchy sequence $\langle f_n \rangle_{n \in \mathbb{N}}$ in $\mathscr{S}(\mathscr{A})$, as in (16). We also know from Theorem 2 that for all ε there exists n sufficiently large so that $\inf_{\nu \in \mathscr{N}} \nu(f_n) > a - \varepsilon$ and $b + \varepsilon > \sup_{\mu \in \mathscr{M}} \mu(f_n)$. Choosing $\varepsilon < (a-b)/2$ we get $\inf_{\nu \in \mathscr{N}} \nu(f_n) > \frac{a+b}{2} > \sup_{\mu \in \mathscr{M}} \mu(f_n)$.

(ii). For each $m \notin \mathscr{M}$ there is then $k_m \in \mathscr{S}(\mathscr{A})$ such that $\sup_{\mu \in \mathscr{M}} \mu(k_m) < m(k_m)$. Let $\mathcal{K} = \{k_m : m \notin \mathscr{M}\}$. For each $k \in \mathcal{K}$ choose one extreme point $\mu_k \in \mathscr{M}$ in the corresponding supporting set of $\mathscr{M}$ and let $\mathscr{M}_0 = \{\mu_k : k \in \mathcal{K}\}$. By construction, each $k \in \mathcal{K}$, when considered as a function on $\mathscr{M}$, attains its maximum on $\mathscr{M}_0$, so that the right hand side of (25) contains $\mathscr{M}$. For each $m \notin \mathscr{M}$ there is $k \in \mathcal{K}$ such that $m(k) > \sup_{\mu \in \mathscr{M}} \mu(k)$ so that the right hand side is included in $\mathscr{M}$. □

It is well known that, combining the Theorems of Eberlein Smulian and of Mazur, and taking convex combinations one may transform a weakly convergent sequence in a Banach space into a norm convergent one. The following result establishes a weak form of this fundamental result which holds even in the absence of weak convergence. The proof exploits some of the ideas introduced by Komlós [8].

Theorem 5. *Let $\langle \lambda_n \rangle_{n \in \mathbb{N}}$ be a sequence in $ba(\mathscr{A})_+$ with $\liminf_n \|\lambda_n\| < \infty$ and let $\Gamma(n) = \text{co}\{\lambda_n, \lambda_{n+1}, \ldots\}$ and $\lambda = \sum_n 2^{-n} \frac{\lambda_n}{1 \vee \|\lambda_n\|}$. There exist $\xi \in ba(\mathscr{A}, \lambda)_+$ and a sequence $\langle \mu_n \rangle_{n \in \mathbb{N}}$ with $\mu_n \in \Gamma(n)$ such that $\langle \mu_n \wedge k\lambda \rangle_{n \in \mathbb{N}}$ converges in norm to $\xi \wedge k\lambda$ for all $k \in \mathbb{R}_+$.*

Proof. There is no loss of generality in passing to a subsequence and assume that the sequence $\langle \lambda_n \rangle_{n \in \mathbb{N}}$ is norm bounded. The set $\Gamma(n,k) = \text{co}\{\mu \wedge k\lambda : \mu \in \Gamma(n)\}$ is convex and, by Theorem 1, relatively weakly compact because uniformly monotone continuous. Of course, $\overline{\Gamma(n,k)} = \overline{\Gamma(n,k)}^w$ is a weakly compact, convex subset of $ba(\mathscr{A})$, [6, V.2.4,V.3.13]. Define $\phi_k : \mathscr{S}(\mathscr{A}) \to \mathbb{R}$ implicitly by letting

$$\phi_k(f) = \lim_n \sup_{\mu \in \overline{\Gamma(n,k)}} \mu(f) \qquad f \in \mathscr{S}(\mathscr{A})$$

Fix $\xi_0 = 0$ and choose $\xi_1 \in ba(\mathscr{A})$ such that $\xi_1 \leq \phi_1$ on $\mathscr{S}(\mathscr{A})$ and $\xi_1(\Omega) = \phi_1(\Omega)$. Assume that for some $k > 1$, it is possible to find $\xi_1, \ldots, \xi_{k-1} \in ba(\mathscr{A})$ such that

$$\xi_0 \leq \xi_1 \leq \ldots \leq \xi_i \leq \phi_i \quad \text{and} \quad \xi_i(\Omega) = \lim_n \sup_{\{\mu \in \overline{\Gamma(n,i)}: \mu \geq \xi_{i-1}\}} \mu(\Omega) \tag{26}$$

for $i = 1, \ldots, k-1$. Let

$$\chi_k(f) \equiv \inf_{\{g \in \mathscr{S}(\mathscr{A}): g \geq f\}} (\phi_k - \xi_{k-1})(g) \qquad f \in \mathscr{S}(\mathscr{A})$$

Since χ_k is a convex functional, we can choose $\eta_k \in ba(\mathscr{A})$ such that $\eta_k(f) \leq \chi_k(f)$ for all $f \in \mathscr{S}(\mathscr{A})$, with equality if $f = \mathbf{1}_\Omega$. Let $\xi_k = \xi_{k-1} + \eta_k$. The inequality $\chi_k(f) \leq 0$ for $f \leq 0$ induces the conclusion $\xi_{k-1} \leq \xi_k \leq \phi_k$. Applying Sion's minimax Theorem repeatedly, we get

$$\begin{aligned}
\inf_{\{g \in \mathscr{S}(\mathscr{A}): g \geq 1\}} \sup_{\mu \in \overline{\Gamma(n,k)}^w} (\mu - \xi_{k-1})(g) &= \sup_{\mu \in \overline{\Gamma(n,k)}^w} \inf_{\{g \in \mathscr{S}(\mathscr{A}): g \geq 1\}} (\mu - \xi_{k-1})(g) \\
&= \sup_{\{\mu \in \overline{\Gamma(n,k)}^w : \mu \geq \xi_{k-1}\}} \inf_{\{g \in \mathscr{S}(\mathscr{A}): g \geq 1\}} (\mu - \xi_{k-1})(g) \\
&= \inf_{\{g \in \mathscr{S}(\mathscr{A}): g \geq 1\}} \sup_{\{\mu \in \overline{\Gamma(n,k)}^w : \mu \geq \xi_{k-1}\}} (\mu - \xi_{k-1})(g) \\
&= \sup_{\{\mu \in \overline{\Gamma(n,k)}^w : \mu \geq \xi_{k-1}\}} (\mu - \xi_{k-1})(\Omega)
\end{aligned}$$

The second line follows from the inequality

$$0 \leq \chi_k(\Omega) \leq \inf_{\{g \in \mathscr{S}(\mathscr{A}): g \geq 1\}} \sup_{\mu \in \overline{\Gamma(n,k)}^w} (\mu - \xi_{k-1})(g)$$

so that, computing the supremum, we may restrict attention to those $\mu \in \overline{\Gamma(n,k)}^w$ such that $\inf_{\{g \in \mathscr{S}(\mathscr{A}): g \geq 1\}} (\mu - \xi_{k-1})(g) > -\infty$ i.e. to the set $\{\mu \in \overline{\Gamma(n,k)}^w : \mu \geq \xi_{k-1}\}$ which is thus convex, weakly compact and non empty. We thus conclude that

$$\begin{aligned}
\xi_k(\Omega) &= \xi_{k-1}(\Omega) + \chi_k(\Omega) \\
&= \xi_{k-1}(\Omega) + \inf_{\{g \in \mathscr{S}(\mathscr{A}): g \geq 1\}} \lim_n \sup_{\mu \in \overline{\Gamma(n,k)}^*} (\mu - \xi_{k-1})(g) \\
&= \xi_{k-1}(\Omega) + \lim_n \inf_{\{g \in \mathscr{S}(\mathscr{A}): g \geq 1\}} \sup_{\mu \in \overline{\Gamma(n,k)}^*} (\mu - \xi_{k-1})(g) \\
&= \xi_{k-1}(\Omega) + \lim_n \sup_{\{\mu \in \overline{\Gamma(n,k)}^w : \mu \geq \xi_{k-1}\}} (\mu - \xi_{k-1})(\Omega) \\
&= \lim_n \sup_{\{\mu \in \overline{\Gamma(n,k)}: \mu \geq \xi_{k-1}\}} \mu(\Omega)
\end{aligned}$$

and, by induction, that there exists a sequence $\langle \xi_k \rangle_{k \in \mathbb{N}}$ meeting (26). Corollary 6.(i) and the inequality $\xi_k \leq \phi_k$ imply $\xi_k \in \bigcap_n \overline{\Gamma(n,k)} \subset \bigcap_n \overline{\mathcal{C}(n)}$, where $\mathcal{C}(n) = (\Gamma(n) - ba(\mathscr{A})_+) \cap ba(\mathscr{A})_+$. If ξ denotes the norm limit of the monotonic sequence $\langle \xi_k \rangle_{k \in \mathbb{N}}$, then also $\xi \in \bigcap_n \overline{\mathcal{C}(n)}$. Let $\mu_n \in \Gamma(n)$ and $\delta_n \in ba(\mathscr{A})_+$ be such that, $\nu_n = \mu_n - \delta_n \geq 0$ and that the sequence $\langle \nu_n \rangle_{n \in \mathbb{N}}$ converges to ξ in

norm and let $\gamma_k = \mathrm{LIM}_n\, \mu_n \wedge k\lambda$. Given that $\mu_n \geq \xi_n - |\nu_n - \xi_n|$ and that $\gamma_k(f) \leq \sup_{\mu\in\Gamma(n,k)} \mu(f)$ for all $f \in \mathscr{S}(\mathscr{A})$ we conclude

$$\gamma_k \geq \xi_k \quad \text{and} \quad \gamma_k \in \bigcap_n \overline{\Gamma(n,k)} \tag{27}$$

Moreover, $(\delta_n \wedge k\lambda) - |\xi - \nu_n| \leq (\mu_n - \xi \wedge k\lambda) \wedge k\lambda \leq \mu_n \wedge 2k\lambda - \xi \wedge k\lambda \leq \mu_n \wedge 2k\lambda - \xi_k$. It then follows that

$$\begin{aligned}
\liminf_n \|\delta_n \wedge k\lambda\| &\leq \lim_k \mathop{\mathrm{LIM}}_n (\delta_n \wedge k\lambda)(\Omega) \\
&\leq \lim_k \{\mathop{\mathrm{LIM}}_n (\mu_n \wedge 2k\lambda)(\Omega) - \xi_k(\Omega)\} \\
&= \lim_k \gamma_{2k}(\Omega) - \xi(\Omega) \\
&\leq \lim_k \lim_n \sup_{\{\mu\in\overline{\Gamma(n,2k)}:\mu\geq\xi_{2k-1}\}} \mu(\Omega) - \xi(\Omega) && \text{by (27)} \\
&= \lim_k \xi_{2k}(\Omega) - \xi(\Omega) && \text{by (26)} \\
&= 0
\end{aligned}$$

Moving to a subsequence if necessary, we conclude that $\lim_n \|\mu_n \wedge k\lambda - \xi \wedge k\lambda\| = \lim_n \|\mu_n \wedge k\lambda - \nu_n \wedge k\lambda\| \leq \lim_n \|\delta_n \wedge k\lambda\| = 0$, as claimed $\square$

It is clear from the proof that the condition $\liminf_n \|\lambda_n\| < \infty$ may be replaced with the inequality $\lim_k \lim_n \sup_{\mu\in\Gamma(n,k)} \|\mu\| < \infty$, which is more general but less perspicuous.

One should also remark that if the sequence $\langle\lambda_n\rangle_{n\in\mathbb{N}}$ in Theorem 5 is weakly convergent, then, by the uniform absolute continuity property, $\mu_n \wedge k\lambda$ converges (in norm) to μ_n uniformly in $n \in \mathbb{N}$ and thus the sequence $\langle\mu_n\rangle_{n\in\mathbb{N}}$ converges strongly to ξ. Theorem 5 is then indeed a generalization of more classical results. Some implications of Theorem 5 are developed in [4].

6. The Halmos-Savage Theorem and its Implications

The results of the preceding section mainly develop the orthogonality implications of Lemma 1. We may as well deduce interesting conclusions concerning absolute continuity, among which the following finitely additive version of the Lemma of Halmos and Savage [7, Lemma 7, p. 232].

Theorem 6 (Halmos and Savage)**.** *$\mathscr{M} \subset ba(\mathscr{A}, \lambda)$ if and only if $\mathscr{M} \subset ba(\mathscr{A}, m)$ for some $m \in \mathbf{A}(\mathscr{M})$.*

Proof. λ dominates $\mathscr{M}$ if and only if $\lambda^c_{\mathscr{M}}$ does. The claim follows from Lemma 1. $\square$

As is well known, Halmos and Savage provided applications of this result to the theory of sufficient statistics. Another possible development is the following finitely additive version of a well known Theorem of Yan [10, Theorem 2, p. 220]:

Corollary 7 (Yan)**.** *Let $\mathcal{K} \subset L^1(\lambda)$ be convex with $0 \in \mathcal{K}$, $\mathcal{C} = \mathcal{K} - \mathscr{S}(\mathscr{A})_+$ and denote by $\overline{\mathcal{C}}$ the closure of $\mathcal{C}$ in $L^1(\lambda)$. The following are equivalent:*

(i) for each $f \in L^1(\lambda)_+$ with $|\lambda|(f) > 0$ there exists $\eta > 0$ such that $\eta f \notin \overline{\mathcal{C}}$;
(ii) for each $A \in \mathscr{A}$ with $|\lambda|(A) > 0$ there exists $d > 0$ such that $d\mathbf{1}_A \notin \overline{\mathcal{C}}$;
(iii) there exists $m \in \mathbb{P}_{ba}(\mathscr{A})$ such that (a) $\mathcal{K} \subset L^1(m)$ and $\sup_{k\in\mathcal{K}} m(k) < \infty$, (b) $m \in ba_\infty(\mathscr{A}, \lambda)$ and (c) $m(A) = 0$ if and only if $|\lambda|(A) = 0$.

Proof. The implication $(i)\Rightarrow(ii)$ is obvious. If A and d are as in (ii) there exists a continuous linear functional ϕ^A on $L^1(\lambda)$ separating $\{d\mathbf{1}_A\}$ and $\overline{\mathcal{C}}$ and ϕ^A admits the representation $\phi^A(f) = \mu^A(f)$ for some $\mu^A \in ba_\infty(\mathscr{A}, \lambda)$ such that $\mu^A \le c^A|\lambda|$, Corollary 4. Thus $\sup_{h\in\mathcal{C}} \mu^A(f) \le a < b < d\mu^A(A)$. The inclusion $0 \in \mathcal{C}$ implies $a \ge 0$ so that $\mu^A(A) > 0$; moreover, $\mu^A \ge 0$ as $-\mathscr{S}(\mathscr{A})_+ \subset \mathcal{C}$. By normalization we can assume $\|\mu^A\| \vee c^A \vee a \le 1$. The collection $\mathscr{M} = \{\mu^A : A \in \mathscr{A},\ |\lambda|(A) > 0\}$ so obtained is dominated by λ and therefore by some $m \in \mathbf{A}(\mathscr{M})$, by Theorem 6. Thus $m \le |\lambda|$, $\|m\| \le 1$ and $\sup_{h\in\mathcal{C}} m(h) \le 1$. If $A \in \mathscr{A}$ and $|\lambda|(A) > 0$ then $m \gg \mu^A$ implies $m(A) > 0$. By normalization we can take $m \in \mathbb{P}_{ba}(\mathscr{A})$. Let m be as in (iii) so that $L^1(\lambda) \subset L^1(m)$. If $f \in L^1(\lambda)_+$ and $|\lambda|(f) > 0$ then $f \wedge n$ converges to f in $L^1(\lambda)$ [6, III.3.6] so that we can assume that f is bounded. Then, by [2, 4.5.7 and 4.5.8] there exists an increasing sequence $\langle f_n\rangle_{n\in\mathbb{N}}$ in $\mathscr{S}(\mathscr{A})$ with $0 \le f_n \le f$ such that f_n converges to f in $L^1(\lambda)$ and therefore in $L^1(m)$ too. For n large enough, then, $|\lambda|(f_n) > 0$ and, f_n being positive and simple, $m(f_n) > 0$. But then $m(f) = \lim_n m(f_n) > 0$ so that ηf cannot be an element of $\overline{\mathcal{C}}$ for all $\eta > 0$ as $\sup_{h\in\overline{\mathcal{C}}} m(h) < \infty$. □

An application of Corollary 7 is obtained in [4].

One may also draw from Theorem 6 some implications on the structure of a finitely additive set function.

Theorem 7. *Let $\mathscr{M} \subset ba(\mathscr{A}, \lambda)$ and let $\mathscr{H}_0 \subset \mathscr{A}$ generate the ring $\mathscr{H}$. There exist $H_1, H_2, \ldots \in \mathscr{H}_0$ such that, letting $G_n = H_n \backslash \bigcup_{k<n} H_k$ and $G = \bigcap_n H_n^c$, the following holds:*

$$|\mu|^*(H \cap G) = 0 \quad \text{and} \quad \mu(A \cap H) = \sum_n \mu\left(A \cap H \cap G_n\right) \qquad \mu \in \mathscr{M},\ A \in \mathscr{A},\ H \in \mathscr{H} \tag{28}$$

Moreover: (i) if $\mu \in \mathscr{M}$ is $\mathscr{H}_0$-inner regular then

$$\mu(A) = \sum_n \mu\left(A \cap G_n\right) \qquad A \in \mathscr{A} \tag{29}$$

(ii) if $\mathscr{H}_0$ is closed with respect to countable unions then

$$\mu(A) = \mu(A \cap G) + \sum_n \mu(A \cap G_n) \qquad \mu \in \mathscr{M},\ A \in \mathscr{A} \tag{30}$$

Proof. With no loss of generality, let $\lambda \ge 0$ and write $\mathscr{M} = \{\lambda_H : H \in \mathscr{H}_0\}$. By Theorem 6, choose $m_0 = \sum_n \alpha_n \lambda_{H_n} \in \mathbf{A}(\mathscr{M})$ to be such that $m_0 \gg \mathscr{M}$. Let G and G_n be as in the statement and define $m = \sum_n \lambda_{G_n}$. Observe that $m \ge m_0$ and that, by construction, $\lim_k m(\bigcap_{n<k} H_n^c) = 0$. But then, for each $H \in \mathscr{H}_0$ we conclude $\lim_k \lambda_H(\bigcap_{n<k} H_n^c) = \lim_k \lambda(H \cap \bigcap_{n<k} H_n^c) = 0$ and, by absolute continuity, $|\mu|^*(H \cap G) \le \lim_k |\mu|(H \cap \bigcap_{n<k} H_n^c) = 0$ for all $\mu \in \mathscr{M}$. Consequently, if

$A \in \mathscr{A}$ and $H \in \mathscr{H}_0$

$$\begin{aligned}\mu(A \cap H) &= \mu\left(A \cap H \cap \left(\bigcup_{n<k} G_n \cup \bigcap_{n<k} G_n^c\right)\right) \\ &= \lim_k \mu\left(A \cap H \cap \bigcup_{n<k} G_n\right) \\ &= \sum_n \mu(A \cap H \cap G_n)\end{aligned}$$

The set function $\sum_n \mu_{G_n}$ agrees with μ on the ring $\mathscr{R}$ consisting of all finite, disjoint unions of sets of the form $A \cap H$ with $A \in \mathscr{A}$ and $H \in \mathscr{H}_0$. Another ring is the collection $\mathscr{J} = \{H \in \mathscr{H} : A \cap H \in \mathscr{R} \text{ for all } A \in \mathscr{A}\}$ which therefore coincides with $\mathscr{H}$. Thus, $\{H \cap A : H \in \mathscr{H},\ A \in \mathscr{A}\} \subset \mathscr{R}$ which proves (28).

If $\mu \in \mathscr{M}$ is $\mathscr{H}_0$-inner regular, then,

$$\mu^+(A) = \sup_{\{H \in \mathscr{H}_0 : H \subset A\}} \mu(H) = \sup_{\{H \in \mathscr{H}_0 : H \subset A\}} \sum_n \mu(H \cap G_n) \leq \sum_n \mu^+(A \cap G_n) \leq \mu^+(A)$$

the last inequality following from additivity. Exchanging μ with $-\mu$ proves (29). Eventually, if $\mathscr{H}_0$ is closed with respect to countable unions, then $\bigcup_{n>k} G_n \in \mathscr{H}$ and, by (28), $\mu\left(\bigcup_{n>k} G_n\right) = \sum_{n>k} \mu(G_n)$ from which (30) readily follows. $\square$

The following Corollary 8 illustrates a special case.

Corollary 8. *Let Ω be a separable metric space, $\mathscr{A}$ its Borel σ-algebra and $\mathscr{M} \subset ca(\mathscr{A}, \lambda)$. If π is a partition of Ω into open sets then there exist $H_1, H_2, \ldots \in \pi$ such that*

$$\mu(A) = \sum_n \mu(A \cap H_n) \qquad A \in \mathscr{A},\ \mu \in \mathscr{M} \tag{31}$$

Proof. Under the current assumptions, for each increasing net $\langle O_\alpha \rangle_{\alpha \in \mathfrak{A}}$ of open sets we have $\lambda(\bigcup_\alpha O_\alpha) = \lim_\alpha \lambda(O_\alpha)$, [3, Proposition 7.2.2]. Let $\mathscr{H}_0 = \pi$ extract $H_1, H_2, \ldots \in \pi$ as in Theorem 7 and observe that, π being a partition, $G_n = H_n$ for $n = 1, 2, \ldots$; moreover $G = \bigcup_{H \in \pi, H \subset G} H$ and so $\lambda(G) = 0$. We conclude that (31) holds. $\square$

To motivate further our interest in the preceding conclusions, assume that π is an $\mathscr{A}$ partition and that λ is π-inner regular. Then for each $H \in \pi$ and $A \in \mathscr{A}$ one may define $\sigma(A|H) = \lambda(A \cap H_n)/\lambda(H_n)$ if $H = H_n$ and $\lambda(H_n) \neq 0$ or $\sigma(A|H) = m_H(A)$ for any $m_H \in \mathbb{P}_{ba}(\mathscr{A})$ with $m_H(H) = 1$. Write $\sigma(A|\pi) = \sum_{H \in \pi} \sigma(A|H)\mathbf{1}_H$. Then,

$$\lambda(A) = \int \sigma(A|\pi) d\lambda \qquad A \in \mathscr{A} \tag{32}$$

This follows from $\int \sigma(A|\pi) d\lambda = \sum_n \sigma(A|G_n)\lambda(G_n) + \int_G \sigma(A|\pi) d\lambda = \sum_n \lambda(A \cap G_n) = \lambda(A)$. In the terminology introduced by Dubins [5], λ is then strategic along any partition relatively to which it is inner regular.

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